On the Solutions of the Geodetic Exterior/Interior Fixed Boundary Value Problems

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Overview

• Background
• The exterior and interior boundary value problem (BVP)
• Harmonic and non-harmonic solution of the interior BVP inside the topographic mass
• Non-harmonic solution for constant density of topographic mass
• Conclusions
Background

• cm-geoid accuracy requires geoid computation theory being accurate to mm

• Approximations in current geoid computation theories:
  Linearization of the BVP
  Taylor expansions in the gravity/topographic reduction

Geoid computed using different reductions can differ up to cms
Background (continued)

- Earth’s surface can be considered as known (GPS); gravity disturbances can replace gravity anomalies
- Geoid over land areas can be computed by solving interior BVP (Poisson equation)
- Current solutions of interior BVP requires knowledge of global mass density (impossible task)
- Solutions are sought under the assumption that the topographic mass density is known
Fixed geodetic boundary value problems (BVP)

- The exterior BVP:
\[
\begin{align*}
\Delta W_E(x) &= 2\omega^2 \quad x \in \Omega_E \\
|\nabla W_E(x)| &= g(x) \quad x \in S
\end{align*}
\]

where $S$ is known.

- The interior BVP:
\[
\begin{align*}
\Delta W_I(x) &= -4\pi G \rho(x) + 2\omega^2 \quad x \in \Omega_I \\
|\nabla W_I(x)| &= g(x) \quad x \in S
\end{align*}
\]
Simplification of boundary values

- Replacing the norm of the gravity by the vertical derivatives:

\[ g(x) = \text{grad} W(x) = g_u s_u^0 + g_\theta s_\theta^0 + g_\lambda s_\lambda^0 \]

\[ g_u(x) = \sqrt{g^2 - (g_\theta^2 + g_\lambda^2)} = g(x)\sqrt{1 - \mathcal{G}^2} \]

where \( \mathcal{G}^2 = \xi^2 + \eta^2 \)

- Introducing interior reference potential that satisfies:

\[ \Delta U_I(x) = -4\pi G \rho_{\text{ell}}(x) + 2\omega^2 \]
Exterior/interior BVPs (continued)

• The exterior BVP for disturbing potential

\[
\begin{align*}
\Delta T_E(x) &= 0 \quad x \in \Omega_E \\
\frac{\partial T_E(x)}{\partial u} &= -h_1 \delta g(x) + h_1 O(\vartheta^2(x)) \quad x \in S
\end{align*}
\]

• The interior BVP for disturbing potential

\[
\begin{align*}
\Delta T_I(x) &= -4\pi G \delta \rho(x) \quad x \in \Omega_I \\
\frac{\partial T_I(x)}{\partial u} &= -h_1 \delta g(x) + h_1 O(\vartheta^2(x)) \quad x \in S
\end{align*}
\]
Exterior/interior BVPs (continued)

where

\[ \delta \rho(x) = \begin{cases} 
\rho(x) & x \in \Omega'_l \\
\rho(x) - \rho_{ell}(x) & x \in \Omega_{ell} 
\end{cases} \]

\( \Omega'_l \) is the space between the Earth’s surface and the reference ellipsoid
Solutions of the interior BVP

- Decompose solution in the space of topography into harmonic and non-harmonic

\[ T_I = T^H + T^t \]

which are defined by:

\[
\begin{aligned}
\Delta T^H(x) &= 0 & \quad x \in \Omega'_i \\
\frac{\partial T^H(x)}{\partial u} &= -h_i \delta g + O(h_i^2) & \quad x \in S \\
\Delta T^t(x) &= -4\pi G \rho(x) & \quad x \in \Omega'_i \\
\frac{\partial T^t(x)}{\partial u} &= 0 & \quad x \in S
\end{aligned}
\]
Solutions of the interior BVP

- Harmonic solution can be obtained by using the method of analytical downward continuation
- Non-harmonic potential \( T_i \) satisfies the compatibility condition and the solutions of BVP exist and unique.
Strategy of non-harmonic solutions

• Two step strategy for non-harmonic solutions:

1. Solution of Bouguer sphere:

\[
\begin{align*}
\Delta T_0(x) &= -4\pi G \rho(x) \quad x \in \Omega \\
T_0(x) &= 0, \quad \frac{\partial T_0(x)}{\partial r} = 0 \quad x \in S
\end{align*}
\]

2. Corrector field \( T^C(x) = T_0(x) - T_0(x) \) that satisfies:

\[
\begin{align*}
\Delta T^C(x) &= 0 \quad x \in \Omega'_f \\
T^C(x) &= -\tilde{T}_0(x), \quad \frac{\partial T^C(x)}{\partial r} = -\frac{\partial \tilde{T}_0(x)}{\partial r} \quad x \in S
\end{align*}
\]
A special case: constant density of topography

1. Solution of the Bouguer sphere:

\[ T_0(h_p) \approx -2\pi G \rho h_p^2 \left(1 + \frac{2}{3R} h_p\right) \]

2. BVP of the corrector field:

\[
\begin{cases}
\Delta T^C(x) = 0 & \quad x \in \Omega' \\
T^C(x) = 2\pi G \rho (h_p - h_s)^2 & \quad x \in S
\end{cases}
\]
Conclusions

• Geoid can be computed by solving interior BVP

• Solutions of the interior BVP are decomposed into harmonic and non-harmonic components

• Harmonic solution can be obtained by using analytical downward continuation

• Non-harmonic solution can be split into a solution of Bouguer sphere and the corrector filed

• Other method can be used, e.g. finite element method
Questions?

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