Boundary Element Method

for

the Neumann Geodetic Boundary Value Problem

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**Special thanks to:**

**The High Performance Centre**

**CINECA**

(Interuniversity Consortium) in Bologna, Italy

namely to:

Giovanni Erbacci, Gerardo Ballabio and Fiorella Sgallari
• **Disturbing potential:**

\[ T(x) = W(x) - U(x) \quad x \in \mathbb{R}^3 \]

\[ \nabla T(x) = \nabla W(x) - \nabla U(x) = g(x) - \gamma(x) \]  

(different directions \( \rightarrow \) negligibly small)

\[ \langle \nabla T(x) , n_e(x) \rangle = \langle g(x) , n_e(x) \rangle - \langle \gamma(x) , n_e(x) \rangle \approx -g(x) + \gamma(x) = -\delta g(x) \]

• **Surface gravity disturbance:**

\[ \delta g(x) = g(x) - \gamma(x) \quad x \in \Gamma \]

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**Neumann Geodetic BVP**

\[ \Delta T(x) = 0 \quad x \in \text{ext. } \Omega \]

\[ \langle \nabla T(x) , n_e(x) \rangle = -\delta g(x) \quad x \in \Gamma \]

\[ T(x) \rightarrow 0 \quad x \rightarrow \infty \]

the exterior oblique derivative BVP for the Laplace equation
Surface gravity disturbances - Neumann BC

Surface gravity disturbance

\[ \delta g(P) = g(P) - \gamma(P) \]

 gravitational measurement accompanied by levelling

\[ \Delta g(P) = g(P) - \gamma(Q) \]

(quasi)geoid undulation included in \( \Delta g \)

but in the acceleration units

it yields the Robin problem

Newton BC - fundamental gravimetric equation

!!! but fully dependent on levelling !!!

Earth’s surface

Telluroid

Geoid

Ellipsoid

at the same time: determination of

the fixed boundary

!!! NO LEVELLING !!!
**BEM for the Laplace equation**

**Direct BEM formulation** ⇒ **Boundary integral equation** derived from the Laplace equation by applying Green's second theorem

- **Laplace equation**
  \[ \Delta T(p) = 0 \quad p \in \Omega \]

- **Boundary Integral equation**
  \[ \int_{\Omega} \langle \Delta T(p), w \rangle = 0 \]
  (a base of the weak (integral) formulation)
  (for any weight function \( w \))

- **Green's second theorem** (where \( w = G \))
  \[
  4\pi T(p) + \int_{\Gamma} \frac{\partial G(p, q)}{\partial n_q} T(q) d\Gamma_q = \int_{\Gamma} G(p, q) \frac{\partial T(q)}{\partial n_q} d\Gamma_q
  \]
  \( p, q \in \Gamma \)

**Kernel function in 3D** ⇒ fundamental solution of the Laplace equation

\[
G(p, q) = \frac{1}{4\pi \cdot |p - q|}
\]

**Indirect BEM formulation** ⇒ integral representation of the potential \( T \) by:
- single layer potential (right-hand-side of BIE)
- double layer potential (left-hand-side of BIE)
Collocation with linear basis functions

**Collocation method** ⇒ an appropriate numerical technique
step from the integral equation to the linear system of equations

**Boundary** $\Gamma$ **expressed as a set of panels (elements):** planar triangles (3D)

\[ \Gamma = \sum_{i=1}^{N} \Delta \Gamma_i \]

- $C^0$ - constant elements $c(p) = 1/2$
- $C^1$ - linear elements $c(p) = \varphi (1 - \cos \alpha) / 4\pi$
- $C^2$ - quadratic elements

**Discrete form of the boundary integral equation**
⇒ **collocation with linear basis functions**
the boundary functions $T$ and $\delta g$ represented by linear function on each element

\[
c_i T_i \psi_i + \sum_{j=1}^{N} \int_{\text{supp } \psi_j} \frac{\partial G_{ij}}{\partial n_q} T_j \psi_j d\Gamma_j = \sum_{j=1}^{N} \int_{\text{supp } \psi_j} G_{ij} \delta g_j \psi_j d\Gamma_j
\]

- $c_i$ – “spatial angle”
- $\psi_i$ – linear basis function
- $N$ – number of nodes
**Discretization of integral operators**

**Integral approximations in collocation points**

- **regular integrals** ⇒ **Gaussian quadrature rules for the triangle**

\[
M_{ij} = \frac{1}{4\pi} \sum_{l=1}^{L} A_{jl} k_{ij} \sum_{k=1}^{K} \frac{1}{l_{ik}} \psi_k w_k \\
Q_{ij} = \frac{1}{4\pi} \sum_{l=1}^{L} A_{jl} \cos \alpha_j \sum_{k=1}^{K} \frac{1}{l_{ik}} \psi_k w_k
\]

- **nonregular integrals (singular elements)** ⇒ **a special treatment**
  (because of the singularity of the kernel function)

\[
M_{ii} = c_i
\]

(in case of the planar elements)

\[
Q_{ii} = \frac{1}{2\pi} \sum_{s=1}^{S} A_{is} \ln \frac{\beta_s + \alpha_s}{2} \theta \left( \frac{\beta_s}{2} \right)
\]

the analytical form thanks the weak singularity of \( G \)
**Linear system of equations**

- Linear system of equations
  - Opportunity to choose type of BC
    - Neumann BC – land & continents
    - Dirichlet BC – oceans & seas

<table>
<thead>
<tr>
<th>Type of BC</th>
<th>Neumann BC</th>
<th>Mixed BC</th>
</tr>
</thead>
</table>
| properties of the system matrix | • dense, nonsymmetric, positive-definite  
• strict diagonal-dominant  
• generally well conditioned | • dense, nonsymmetric, positive-definite  
• not diagonal-dominant  
• worse conditioned |

**Linear solver:**

- BiConjugate Gradient Stabilized Method (Nonstationary iterative method)
  - BiCGSTAB

**Final transformation:**

- surface disturbing potential
  - height anomalies

\[ M \cdot x = f \]

\[ M \cdot T = Q \cdot \delta g \]

if \( W(P) = U(Q) \)

\[ \zeta(p) \approx \frac{T(p)}{\gamma(p)} \]
**Linear system of equations**

**Code parallelization:**

\[ M \cdot x = f \]

**Message Passing Interface (MPI)**

- computing on several processors
- computing extension
- more precise results

**Computations performed**

at the High Performance Centre

![CINECA](logo)

Consorzio Interuniversitario in Bologna, Italy

**on the parallel supercomputer**

IBM SP5

1216 GB RAM

**The current configuration**

- 60 nodes with 8 processors (16 GB per node)
- 4 nodes with 8 processors (124 GB per node)
**Numerical experiment - global solutions**

Global triangulation of topography

Ellipsoidal (geodetic) heights
GTOPO-30 + EGM-96

Variational solution
(BEM applied to NGBVP)
should converge to EGM-96

Surface gravity disturbances
generated from geopotential coefficients of EGM-96
(using program f477s)

- Surface gravity disturbances generated from geopotential coefficients of EGM-96 (using program f477s)
- Global triangulation of topography
- Ellipsoidal (geodetic) heights
  - GTOPO-30 + EGM-96
- Variational solution
  - (BEM applied to NGBVP)
  - should converge to EGM-96
- confirms mathematical reliability
Global quasigeoid model

3D BEM applied to Neumann geodetic BVP
collocation with linear basis functions (124 418 nodes)
Accuracy of the variational solution

Profiles along the parallel of latitude: N30°

- **Nodes**: 1,946
  - **MEAN**: 0.580 m
  - **ST. DEV.**: 7.525 m
  - **MAX**: 47.875 m
  - **MIN**: -42.496 m

- **Nodes**: 5,402
  - **MEAN**: -0.349 m
  - **ST. DEV.**: 3.951 m
  - **MAX**: 32.778 m
  - **MIN**: -33.064 m

- **Nodes**: 38,402
  - **MEAN**: 0.457 m
  - **ST. DEV.**: 0.869 m
  - **MAX**: 7.987 m
  - **MIN**: -7.368 m

- **Nodes**: 44,378
  - **MEAN**: -0.023 m
  - **ST. DEV.**: 0.692 m
  - **MAX**: 6.163 m
  - **MIN**: -6.283 m

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VI Hotine-Marussi Symposium 29 May - 2 June, Wuhan, PR China
### Comparison with EGM-96

#### Statistics

<table>
<thead>
<tr>
<th></th>
<th>N</th>
<th>44 378</th>
<th>48 602</th>
<th>60 002</th>
<th>124 418</th>
</tr>
</thead>
<tbody>
<tr>
<td>Δφ</td>
<td>1.047°</td>
<td>1°</td>
<td>0.9°</td>
<td>0.625°</td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>-0.023 m</td>
<td>-0.373 m</td>
<td>-0.258 m</td>
<td>-0.286 m</td>
<td></td>
</tr>
<tr>
<td>Max.</td>
<td>6,163 m</td>
<td>4,611 m</td>
<td>2,869 m</td>
<td>1,428 m</td>
<td></td>
</tr>
<tr>
<td>Min.</td>
<td>-6,283 m</td>
<td>-5,480 m</td>
<td>-4,062 m</td>
<td>-3,285 m</td>
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<tr>
<td>St.dev.</td>
<td>0.692 m</td>
<td>0.583 m</td>
<td>0.293 m</td>
<td>0.203 m</td>
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</table>

#### Computational aspects

<table>
<thead>
<tr>
<th>Code</th>
<th>serial</th>
<th>MPI</th>
<th>MPI</th>
<th>MPI</th>
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<tbody>
<tr>
<td>Processors</td>
<td>1</td>
<td>12</td>
<td>16</td>
<td>80</td>
</tr>
<tr>
<td>Blocking</td>
<td>-</td>
<td>unlimited</td>
<td>unlimited</td>
<td>8</td>
</tr>
<tr>
<td>Iterations</td>
<td>16</td>
<td>16</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>Matrix assembly</td>
<td>-</td>
<td>1 563 s</td>
<td>1 731 s</td>
<td>1 375 s</td>
</tr>
<tr>
<td>BiCGSTAB time</td>
<td>-</td>
<td>181 s</td>
<td>119 s</td>
<td>116 s</td>
</tr>
<tr>
<td>Total time</td>
<td>-</td>
<td>1 718 s</td>
<td>1 852 s</td>
<td>1 496 s</td>
</tr>
</tbody>
</table>
Comparison with EGM-96

Variational solution ⇔ EGM-96

/ BEM application ⇔ spherical harmonics /

**STATISTICS**

<table>
<thead>
<tr>
<th></th>
<th>Total</th>
<th>Continents</th>
<th>Oceans</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nodes</td>
<td>124 418</td>
<td>40 213</td>
<td>84 205</td>
</tr>
<tr>
<td>Mean</td>
<td>-0.286 m</td>
<td>-0.406 m</td>
<td>-0.228 m</td>
</tr>
<tr>
<td>Max.</td>
<td>1.428 m</td>
<td>0.577 m</td>
<td>1.428 m</td>
</tr>
<tr>
<td>Min.</td>
<td>-3.285 m</td>
<td>-3.285 m</td>
<td>-1.776 m</td>
</tr>
<tr>
<td>St. dev.</td>
<td>0.203 m</td>
<td>0.268 m</td>
<td>0.139 m</td>
</tr>
</tbody>
</table>
**Reduction of memory requirements**

**Main drawback:** enormous requirements for the internal memory storage

**MPI implementation**
- computing on several processors
- elimination of far zones interactions

**Geopotential models**
- Incorporating the known geopotential models
- Kernel function for matrix $M$:
  \[ Q(x,y) = O(R^{-2}) \quad \text{for } R \to \infty \]
- Limit:
  \[ \text{if } |x - y| > \lim \Rightarrow M_{ij} \to f \]
  \Rightarrow \text{sparse system matrix } M

**Fast Multipole Method (FMM)**
FMM approximates interactions by performing multipole expansions at a course resolution reducing interactions evaluation

\[ O(N \times N) \to O(N \log N) \]

domain decomposition scheme \to particles clustering / QuadTrees (2D), Octrees (3D) / \to expansions coefficients (far zones)

\Rightarrow \text{near zones handled directly}

**computing extension** \Rightarrow more accurate results
### Far zones incorporated from EGM-96

#### Comparison with EGM-96

<table>
<thead>
<tr>
<th></th>
<th>N</th>
<th>124 418</th>
<th>124 418</th>
<th>124 418</th>
<th>124 418</th>
<th>124 418</th>
<th>124 418</th>
<th>194 402</th>
<th>375 002</th>
</tr>
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<tbody>
<tr>
<td>( \Delta \varphi ) (size of elements)</td>
<td>0.625°</td>
<td>0.625°</td>
<td>0.625°</td>
<td>0.625°</td>
<td>0.625°</td>
<td>0.625°</td>
<td>0.5°</td>
<td>0.36°</td>
<td></td>
</tr>
<tr>
<td>Far zones Limit</td>
<td>no</td>
<td>A/2</td>
<td>A/3</td>
<td>A/4</td>
<td>A/8</td>
<td>A/8</td>
<td>A/2</td>
<td>A/8</td>
<td></td>
</tr>
<tr>
<td>Gauss q. G7</td>
<td>G7</td>
<td>G7</td>
<td>G7</td>
<td>G7</td>
<td>G7</td>
<td>G7</td>
<td>G3</td>
<td>G3</td>
<td></td>
</tr>
<tr>
<td>Residuals Mean</td>
<td>-0.286 m</td>
<td>-0.457 m</td>
<td>-0.490 m</td>
<td>-0.508 m</td>
<td>-0.538 m</td>
<td>-0.539 m</td>
<td>-0.507 m</td>
<td>-0.614 m</td>
<td></td>
</tr>
<tr>
<td>Max.</td>
<td>1,428 m</td>
<td>1,248 m</td>
<td>1,214 m</td>
<td>1,195 m</td>
<td>1,164 m</td>
<td>1,163 m</td>
<td>0.727 m</td>
<td>0.268 m</td>
<td></td>
</tr>
<tr>
<td>Min.</td>
<td>-3,285 m</td>
<td>-3,446 m</td>
<td>-3,493 m</td>
<td>-3,525 m</td>
<td>-3,594 m</td>
<td>-3,603 m</td>
<td>-4,639 m</td>
<td>-2,657 m</td>
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</tr>
<tr>
<td>St. dev.</td>
<td>0.203 m</td>
<td>0.202 m</td>
<td>0.205 m</td>
<td>0.208 m</td>
<td>0.214 m</td>
<td>0.215 m</td>
<td>0.214 m</td>
<td>0.201 m</td>
<td></td>
</tr>
<tr>
<td>Memory requirements Full M</td>
<td>116 GB</td>
<td>116 GB</td>
<td>116 GB</td>
<td>116 GB</td>
<td>116 GB</td>
<td>116 GB</td>
<td>282 GB</td>
<td>1.05 TB</td>
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<tr>
<td>Sparse M</td>
<td>116 GB</td>
<td>6.0 GB</td>
<td>2.62 GB</td>
<td>1.43 GB</td>
<td>338 MB</td>
<td>338 MB</td>
<td>14.5 GB</td>
<td>3.25 GB</td>
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</tr>
<tr>
<td>% of full matrix</td>
<td>100%</td>
<td>5.17%</td>
<td>2.27%</td>
<td>1.24%</td>
<td>0.32%</td>
<td>0.32%</td>
<td>5.16%</td>
<td>0.31%</td>
<td></td>
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<tr>
<td>Processors</td>
<td>80</td>
<td>8</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>22</td>
<td>24</td>
<td></td>
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<tr>
<td>Iterations</td>
<td>12</td>
<td>6</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>5</td>
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<tr>
<td>Matrix assembly</td>
<td>1 375 s</td>
<td>17 899 s</td>
<td>14 342 s</td>
<td>13 354 s</td>
<td>13 367 s</td>
<td>4 270 s</td>
<td>15 556 s</td>
<td>16 539 s</td>
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<tr>
<td>BiCGSTAB time</td>
<td>116 s</td>
<td>1997 s</td>
<td>18 s</td>
<td>9 s</td>
<td>4 s</td>
<td>5 s</td>
<td>38 s</td>
<td>16 s</td>
<td></td>
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<tr>
<td>CPU time/procs</td>
<td>1 496 s</td>
<td>19 901 s</td>
<td>14 365 s</td>
<td>13 369 s</td>
<td>13 375 s</td>
<td>4 281 s</td>
<td>15 602 s</td>
<td>16 573 s</td>
<td></td>
</tr>
<tr>
<td>Total CPU time</td>
<td>119 680 s</td>
<td>159 208 s</td>
<td>143 650 s</td>
<td>133 690 s</td>
<td>133 750 s</td>
<td>42 810 s</td>
<td>343 244 s</td>
<td>397 752 s</td>
<td></td>
</tr>
</tbody>
</table>

A - semimajor axis of ellipsoid WGS-84  
G7 – 7 Gaussian points (quadrature of the 5-th order)  
G3 – 3 Gaussian points (linear quadrature)
Comparison with EGM-96

\[ \zeta_x = \frac{GM - GM_0}{r \cdot \gamma} - \frac{(W_0 - U_0)}{\gamma} = -0.528 \text{ m} \]

\[ U_0 = 62 \, 636 \, 851.71 \, \text{m}^3\text{s}^{-2} \]

\[ W_0 = 62 \, 636 \, 856.88 \, \text{m}^3\text{s}^{-2} \]

<table>
<thead>
<tr>
<th>STATISTICS</th>
<th>Area</th>
<th>Total</th>
<th>Continents</th>
<th>Oceans</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nodes</td>
<td>375 002</td>
<td>106 361</td>
<td>268 641</td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>-0.614 m</td>
<td>-0.764 m</td>
<td>-0.554 m</td>
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<tr>
<td>Max.</td>
<td>0.268 m</td>
<td>-0.239 m</td>
<td>0.268 m</td>
<td></td>
</tr>
<tr>
<td>Min.</td>
<td>-2.657 m</td>
<td>-2.657 m</td>
<td>-1.875 m</td>
<td></td>
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<tr>
<td>St. dev.</td>
<td>0.201 m</td>
<td>0.274 m</td>
<td>0.121 m</td>
<td></td>
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</tbody>
</table>
Mixed BC - comparison with EGM-96

Subdomains with different BC
Neumann BC – land & continents
Dirichlet BC – oceans & seas

STATISTICS

<table>
<thead>
<tr>
<th></th>
<th>Nodes</th>
<th>land &amp; continents</th>
<th>oceans &amp; seas</th>
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</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Quasideoidal Heights</td>
<td>Gravity disturbances</td>
</tr>
<tr>
<td>Nodes</td>
<td>124 418</td>
<td>40 213</td>
<td>84 205</td>
</tr>
<tr>
<td>MEAN</td>
<td>-0.167 m</td>
<td></td>
<td>0.041 mGal</td>
</tr>
<tr>
<td>MAX</td>
<td>1.050 m</td>
<td></td>
<td>104.7 mGal</td>
</tr>
<tr>
<td>MIN</td>
<td>-3.636 m</td>
<td></td>
<td>-175.9 mGal</td>
</tr>
<tr>
<td>ST. DEV.</td>
<td>0.264 m</td>
<td></td>
<td>4.809 mGal</td>
</tr>
</tbody>
</table>
**Neumann BC – local refinement**

**Local refinement of triangulation**

**INPUT DATA**

- **ellipsoidal (geodetic) heights:**
  - **levelling heights** (gravimetric mapping) +
  - local quasigeoid model SQM98BF
- **surface gravity disturbances:**
  - from original gravity measurements

**Computation aspects**

<table>
<thead>
<tr>
<th>N</th>
<th>89 677</th>
</tr>
</thead>
<tbody>
<tr>
<td>Processors</td>
<td>48</td>
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<tr>
<td>Blocking</td>
<td>8</td>
</tr>
<tr>
<td>Iterations</td>
<td>12</td>
</tr>
<tr>
<td>Matrix assembly</td>
<td>1 187 s</td>
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<tr>
<td>BiCGSTAB time</td>
<td>86 s</td>
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<tr>
<td>Total time</td>
<td>1 279 s</td>
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</tbody>
</table>

VI Hotine-Marussi Symposium

29 May - 2 June, Wuhan, PR China
Local quasigeoid model in Slovakia

Variational solution

3D BEM application to the Neumann geodetic BVP

\[ \Delta B \times \Delta L \Rightarrow 2.0' \times 1.4' \]
Approximately regular triangulation

VI Hotine-Marussi Symposium
29 May - 2 June, Wuhan, PR China
GPS/Levelling test

61 GPS/Levelling points

- CEGRN (Central European Geodynamic Reference Network)
- SGRN (Slovak Geodynamic Reference Network)

### STATISTICS

<table>
<thead>
<tr>
<th>Polynomial fitting</th>
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<th>yes</th>
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</thead>
<tbody>
<tr>
<td>Coefficients</td>
<td>-</td>
<td>6</td>
</tr>
<tr>
<td>Test points</td>
<td>61</td>
<td>61</td>
</tr>
<tr>
<td>Mean</td>
<td>-0.387 m</td>
<td>0.000 m</td>
</tr>
<tr>
<td>Max.</td>
<td>-0.164 m</td>
<td>0.172 m</td>
</tr>
<tr>
<td>Min.</td>
<td>-0.817 m</td>
<td>-0.158 m</td>
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<tr>
<td>St. dev.</td>
<td>0.165 m</td>
<td>0.060 m</td>
</tr>
</tbody>
</table>

Residuals after polynomial fitting

Fitting function (polynom of 2nd order) 6 coefficients
Conclusions

Boundary Element Method applied to geodetic BVP

- high performance computing (MPI)
- elimination of far zones (FMM, ...)
- opportunity of local refinements (triangulation with discrete points)

⇒ EFFICIENT TOOL for global and local gravity field modelling

Neumann BC

Surface gravity disturbances
practical advantage: GPS + gravimetry without levelling (contribution evident in mountainous areas)

Variational solution directly on the Earth’s surface

local refinements of triangulation
more precise solutions inside refined areas

global gravity field modelling based on adaptive refining procedures

⇒ with hope to access original gravity data

Thank you