On the Inversion for Mass (Re)Distribution from Global (Time-Variable) Gravity Field

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Limitation of (time-variable) gravity signal

“You don’t know where it comes from!”

• Low spatial resolution

• Non-uniqueness in inversion

• Sum of all sources
Gravitational Potential Field

- Newton's gravitational law

\[ U(\mathbf{r}) = G \iiint_{V_0} \frac{\rho(\mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|} dV_0 \]

- Addition theorem

\[ 1/|\mathbf{r} - \mathbf{r}_0| = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (2n + 1)^{-1} \left( r_0^n / r^{n+1} \right) Y_{nm}^*(\Omega) Y_{nm}(\Omega_0) \]

→ Multipole expansion of gravity field

\[ U(\mathbf{r}) = G \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{(2n + 1) r^{n+1}} \left[ \iiint_{V_0} \rho(\mathbf{r}_0) r_0^n Y_{nm}(\Omega_0) dV_0 \right] Y_{nm}^*(\Omega) \]
Gravitational Potential Field (Geoid)

• Multipole expansion of Newton’s formula:

\[
U(r) = G \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{(2n+1)r^{n+1}} \left[ \iiint_{V_0} \rho(r_0) r_0^n Y_{nm}(\Omega_0) \, dV_0 \right] Y_{nm}^*(\Omega)
\]

• Conventional expression (satisfying Laplace Eq. in terms of Stokes Coeff.):

\[
U(r, \theta, \lambda) = \frac{GM}{a} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( \frac{a}{r} \right)^{n+1} P_{nm}(\cos \theta) \left( C_{nm} \cos m\lambda + S_{nm} \sin m\lambda \right)
\]

\[
C_{nm} + iS_{nm} = \frac{1}{(2n+1)Ma^n} \iiint_{V_0} \rho(r) r^n Y_{nm}(\Omega) \, dV
\]
3-D Gravitational Inversion

• Multipole expansion

\[ U(\mathbf{r}) = G \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{(2n+1) r^{n+1}} \left[ \iiint_{V_0} \rho(\mathbf{r}_0) r_0^n Y_{nm}(\Omega_0) dV_0 \right] Y_{nm}^*(\Omega) \]

2\(n+1\) (known) multipoles for each degree \(n\)

• Moment expansion

\[ U(\mathbf{r}) = G \sum_{n=0}^{\infty} \sum_{\alpha,\beta,\gamma \geq 0} \frac{(-1)^n}{\alpha! \beta! \gamma!} \left[ \iiint_{V_0} x_0^\alpha y_0^\beta z_0^\gamma \rho(\mathbf{r}_0) dV_0 \right] \frac{\partial^n}{\partial x^\alpha \partial y^\beta \partial z^\gamma} \left( \frac{1}{|\mathbf{r}|} \right) \]

(n+1)(n+2)/2 (unknown) moments for each \(n\)

“Degree of deficiency” of knowledge is \(n(n-1)/2\) for each degree \(n\).
The degree of deficiency as a function of spherical harmonic degree $n$ in the 3-D gravitational inversion.

<table>
<thead>
<tr>
<th>Degree $n$</th>
<th># multipoles $(2n + 1)$</th>
<th># moments $(n+1)(n+2)/2$</th>
<th>Degree of deficiency $n(n-1)/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1 (monopole)</td>
<td>1 (total mass)</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3 (dipole)</td>
<td>3 (center of mass)</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>5 (quadrupole)</td>
<td>6 (inertia tensor)</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>7 (octupole)</td>
<td>10 (3rd moment)</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>15</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>11</td>
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<td>10</td>
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<tr>
<td>6</td>
<td>13</td>
<td>28</td>
<td>15</td>
</tr>
<tr>
<td>100</td>
<td>201</td>
<td>5151</td>
<td>4950</td>
</tr>
</tbody>
</table>
Additional physical/mathematical constraints leading to unique solutions:

- minimum shear energy
- maximum entropy of $\rho$
- minimum norm-2 variance for the lateral distribution

...
2-D gravitational Inversion on a spherical shell $S_0$

$$C_{nm} + iS_{nm} = \frac{a^2}{(2n+1)M} \int \int_{S_0} \sigma(\Omega) Y_{nm}(\Omega) \, d\Omega$$

$$\sigma(\Omega) = \sum_{n,m} \sigma_{nm} Y^*_{nm}(\Omega)$$

$$\sigma_{nm} = \frac{(2n+1)M}{4\pi a^2} \left( C_{nm} + iS_{nm} \right)$$

It is possible to mimic ANY external field by means of some proper surface density on $S_0$. 
For CHANGES due to mass redistribution on $S_0$ (taking into account of loading effect), in Eulerian description:

\[
\Delta C_{nm}(t) + i \Delta S_{nm}(t) = \frac{a^2 (1 + k'_n)}{(2n + 1) M} \int \int \Delta \sigma(\Omega; t) Y_{nm}(\Omega) \, d\Omega
\]

\[
\Delta \sigma_{nm}(t) = \frac{(2n + 1) M}{4\pi a^2 (1 + k'_n)} \left[ \Delta C_{nm}(t) + i \Delta S_{nm}(t) \right]
\]

Unique!

Loading effect “undone”
(near) Surface Mass Transports

- Earth ellipticity \sim \frac{1}{2} \text{ of } \frac{1}{300} \text{ a } \sim 10 \text{ km}
- Atmosphere scale height \sim 10 \text{ km}
- Ocean \sim < 5 \text{ km}
- Land hydrology \sim < \text{ a few km}
- Crustal/topography change \sim < 30 \text{ km}
Nice things about spherical harmonics:

- Wavelength, or spatial resolution, 
  \[ \sim 40,000/2N \text{ km} \]  
  \[ \Rightarrow \]  
  Concept of spectrum
- Altitude attenuation \[ \sim r^{-n+1} \]
- Geoid: Stokes Coeff. \( (C_{nm}, S_{nm}) \)
- Gravity Disturbance: \( (n+1)(C_{nm}, S_{nm}) \)
- Gravity Anomaly: \( (n-1)(C_{nm}, S_{nm}) \)
- Surface Mass Change: \( (2n+1)(\Delta C_{nm}, \Delta S_{nm}) \)
Seasonal phase of surface-mass anomaly according to GRACE

Jan  Feb  March  Apr  May  June  July  Aug  Sep  Oct  Nov  Dec
Conclusions
for the [external gravity => mass density] inversion:

- The 3-D inversion is non-unique (well-known).
- This 3-D non-uniqueness is associated with the radial (depth) dimension.
- Comparing the (spherical harmonic) multipole expansion and the moment expansion => The degree of deficiency in inversion is \( n(n-1)/2 \) for each degree \( n \).
- The 2-D inversion on a spherical shell is unique.
- In terms of spherical harmonics this 2-D uniqueness is convenient and useful in (global) time-variable gravity studies (such as GRACE).