Modeling the time variations of the Earth gravity field by Ensemble Kalman Filter

Zizhan Zhang  Yang Lu   Houtse Hsu

30 May 2006, Wuhan, China
VI Hotine-Marussi Symposium of Theoretical and Computational Geodesy: Challenge and Role of Modern Geodesy

Institute of Geodesy and Geophysics,
Chinese Academy of Sciences, P.R. China
1. Introduction

- GRACE -- much finer spatial resolution and greater accuracy than previously possible.
- Gravity dataset -- Geopotential coefficients ~40 monthly constrained (GFZ, CSR, JPL).

These make it possible to model the time variations of the Earth gravity field.
2. Objectives

- Spatial-temporal variations of Earth Gravity field from spectral analysis.

- Modeling the time variations of the Earth gravity field by Ensemble Kalman Filter
3. Data used and preprocessing (1/2)

- **Data used:** 36 constrained monthly GRACE gravity models by GFZ, 2003/02 -- 2006/01.
3. Data used and preprocessing (2/2)

- **Noise filtering:** The normalized Gauss kernel averaging function at 800 km (Jekeli, 1981; Wahr et. al., 1998).
4. Temporal variations

(1) General expression of Earth gravity field in spherical harmonics (see, e.g., Chao and Gross, 1987)

\[ U(\theta, \phi) = a \sum_{l=0}^{\infty} \sum_{m=0}^{l} \tilde{P}(\cos \theta)[C_{lm} \cos(m\phi) + S_{lm} \sin(m\phi)] \] (1)

(2) Expression of monthly Earth gravity field from GRACE

\[ U(\theta, \phi, t_k) = a \sum_{l=0}^{\infty} \sum_{m=0}^{l} [C_{lm}(t_k) \cos(m\phi) + S_{lm}(t_k) \sin(m\phi)] \tilde{P}(\cos \theta) \] (2)
4. Temporal variations (con.)

(3) Expression of time variable Earth gravity field

\[
U(\theta, \phi, t) = \overline{U}(\theta, \phi) + \dot{U}(\theta, \phi)t + U_p(\theta, \phi, t) + \varepsilon_U
\]  

\[
U(\theta, \phi, t) = \overline{U}(\theta, \phi) + \dot{U}(\theta, \phi)t + a \frac{1}{N} \sum_{l,m} \alpha_{0lm} + a \frac{2}{N} \sum_{n=1}^{N/2} \sum_{l,m} \delta_n \Omega_{nlm} \cos\left(\frac{2\pi n}{N} t + \omega_n\right)
\]
4. Averaged $\sqrt{\sum |FFT|^2}$ of the temporal variations of all global grid point
4. Per-degree amplitude for different periods

![Graph showing per-degree amplitude for different periods with spherical harmonic degree on the x-axis and geoid height in mm on the y-axis. Dashed lines indicate mean and specific months (36, 18, 12, 9.0, 7.2, 6.0, 5.2, 4.5, 4.0, 3.6, 3.3, 3.0, 2.8, 2.6, 2.4, 2.2, 2.1, 2.0 months).]
4. Cumulative amplitude for different periods

![Graph showing cumulative amplitude for different periods]
4. Seasonal amplitude map
Part I: summary

• The annual amplitude is strongest. The semiannual amplitude is also prominent.

• The linear term is inconspicuous (time series too short).
Modeling the time variations of the Earth gravity field by Ensemble Kalman Filter
5. Ensemble Kalman Filter

The Ensemble Kalman Filter (EnKF) was first introduced into data assimilation by Eveasen in 1994. The Ensemble Kalman Smoother (EnKS) was developed by Eveasen & van Leeuwen in 2000.

- Represents error statistics using an ensemble of model states.
- Evolves error statistics by ensemble integrations.
- “Variance minimizing” analysis scheme operating on the ensemble.
5. Ensemble Kalman Filter

**Scheme of standard KF**
1. Kalman gain computation
   \[ K_e = P_e^f H^T (H P_e^f H^T + R)^{-1} \]
2. Error covariance of analysis
   \[ P_a = (I - K_e H) P_f \]
3. State analysis
   \[ X_a = X_b + K_e (D - HX_b) \]
4. State forecast
   \[ X_b^f = MX_a^{t-1} \]
5. Error covariance forecast
   \[ P_f^f = MP_a M^T + Q \]

**Scheme of EnKF**
1. Error covariance analysis
   \[ P_f^f \approx P_f = \left( \psi^f - \bar{\psi} \right) \left( \psi^f - \bar{\psi} \right)^T \]
   \[ P_a^f \approx P_a = \left( \psi^a - \bar{\psi} \right) \left( \psi^a - \bar{\psi} \right)^T \]
2. Kalman gain computation
   \[ K_e = P_e^f H^T (H P_e^f H^T + R)^{-1} \]
3. State analysis
   \[ X_a = X_b + K_e (D - HX_b) \]
4. State forecast
   \[ X_b^f = MX_a^{t-1} \]

VS
The error covariance matrix

Define ensemble covariances around the ensemble mean

\[ P^f \simeq P^f_e = (\psi^f - \overline{\psi^f})(\psi^f - \overline{\psi^f})^T \]
\[ P^a \simeq P^a_e = (\psi^a - \overline{\psi^a})(\psi^a - \overline{\psi^a})^T \]

- The ensemble mean \( \overline{\psi} \) is the best-guess.
- The ensemble spread defines the error variance.
- The covariance is determined by the smoothness of the ensemble members.
- A covariance matrix can be represented by an ensemble of model states (not unique).
Dynamical evolution of error statistics

Each ensemble member evolve according to the model dynamics which is expressed by a stochastic differential equation

$$d\psi = f(\psi)dt + g(\psi)d\mathbf{q}.$$ 

The probability density then evolve according to Kolmogorov’s equation

$$\frac{\partial \phi}{\partial t} + \sum_i \frac{\partial (f_i \phi)}{\partial \psi_i} = \frac{1}{2} \sum_{i,j} \frac{\partial^2 \phi (gQg^T)_{ij}}{\partial \psi_i \partial \psi_j}.$$ 

This is the fundamental equation for evolution of error statistics and can be solved using Monte Carlo methods.
5. EnKF & EnKS solution procedure

Special case of EnKS.
Only update state at data times
EnKF forms a prior for the EnKS.
6. Equation of model and model errors

Introduces poorly known parameter $\beta_k$ in model

$$
\begin{pmatrix}
q_k \\
\beta_k \\
\psi_k
\end{pmatrix}
= 
\begin{pmatrix}
\alpha q_{k-1} \\
\beta_{k-1} \\
\psi_{k-1} + (1 + \beta_k) \Delta t + \sqrt{\Delta t} \sigma \rho q_k
\end{pmatrix}
+ 
\begin{pmatrix}
\sqrt{1 - \alpha^2} w_{k-1} \\
0 \\
0
\end{pmatrix}
$$

$\beta_k$ is bias of linear trend.

$\alpha$ is time decorrelation of the stochastic forcing.

$\sigma$ is the standard deviation of the model error.

$\rho$ is a factor, which can be determined by

$$
\rho^2 = \frac{1}{\Delta t} \frac{(1 - \alpha^2)}{n - 2\alpha - n\alpha^2 + 2\alpha^{n+1}}
$$

$n$ is the number of time steps.
7. Primary results (EnKF & EnKS) for C30

\[ Y: (C30_t - \text{mean}(C30)) \times 1.0E+10 \]
7. Primary results (EnKS) for C20

\[ Y = (C20_t - \text{mean}(C20)) \times 1.0 \times 10^10 \]
7. Primary results (EnKS) for C21

\[ Y: \left( C21_t - \text{mean}(C21) \right) \times 1.0 \times 10^{11} \]
7. Primary results (EnKS) for S21

Y: \((S21_t - \text{mean}(S21)) \times 1.0 \times 10^{11}\)
7. Primary results (EnKS) for C91

$$Y: (C91_t - \text{mean}(C91)) \times 1.0E+11$$
7. Primary results (EnKS) for S91

\[
Y: (S91_t - \text{mean}(S91)) \times 1.0E+11
\]
Part II: summary

- The EnKS is efficient in modelling the time variable gravity signal.

- The EnKS may be used for
  - bias (e.g. linear trend) and parameter estimation of the dynamical model.
  - noise reduction.

- Data from SLR, Earth rotation, GPS etc. may be assimilated into GRACE data to enhance coefficient estimation by EnKS (EnKF).
Thank you!